Bifurcation Theory and Climate Tipping in the Budyko–Sellers Energy Balance Climate Model

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1 Introduction

1.1 Background

One of the earliest and most notable quantitative climate models in history dates back to 1896, where Professor Svante Arrhenius [2] formalized the idea that the Earth's temperature is determined by a balance between incoming and outgoing energy. However, the late 1960s marked the real pivotal period in the development of Energy Balance Climate Models (EBCM), led by two seminal papers produced by climatologists Mikhail Budyko [3] and William Sellers [13]. Though independently published in 1969, each paper presents latitudinally dependent climate models based on the energy balance of the Earth–atmosphere system and have been frequently cited together as Budyko–Sellers ever since. Budyko–Sellers' EBCM adds a layer of complexity to earlier models by incorporating latitude-dependent temperature, ice-albedo feedback, and diffusive heat transport across latitudes.

Scientists have used these papers to propel further research in climate dynamics. For example, Gerard North redefined the Budyko–Sellers model by writing it explicitly as a diffusion equation in his paper "Theory of Energy-Balance Climate Models" (1975) [10]. North's modified model allows the same tools from partial differential equations to be applied to Budyko–Sellers, making analysis of equilibrium states and tipping behavior more approachable. Herman Held and Thomas Schneider (1994) directly applied Budyko–Sellers to build an ice-line model, using temperature to figure out where ice forms and how albedo changes. Recently, the field has seen developments in climate tipping point research, specifically Arctic sea ice tipping behavior. As a widely applicable tool in the field of climate modeling, the Budyko–Sellers model enables deeper understanding of whether climate change could lead to abrupt, irreversible shifts. While the full Budyko–Sellers framework is latitudinally dependent, this paper focuses on the zero-dimensional reduction, which retains the essential nonlinear ice-albedo feedback and bifurcation structure while remaining at the level of ordinary differential equations.

1.2 Connection to Course Material

This paper builds on the course's limited coverage of bifurcation theory, recalling concepts such as qualitative analysis of first-order autonomous ordinary differential equations, equilibria, stability, and how changes in parameters affect system behavior through phase line analysis. By applying these tools to a nonlinear energy balance model for Earth's temperature, where equilibria correspond to steady climate states, we introduce more complex behavior of nonlinear feedback loops corresponding to temperature-dependent albedo, including saddle-node bifurcations and hysteresis. Therefore, we are able to extend the course framework to analyze climate tipping.

1.3 Why Mathematical Modeling?

Mathematical modeling allows relevant climate problems to be integrated with theoretical physics. Such concrete equations not only allow representation of fundamental laws of thermodynamics but also enable researchers to test hypotheses and forecast long-term trends. Both applications are essential to safely simulate the climate's response to various external forcings.

2 Statement of the Problem

In this report, we will describe the derivation of the Budyko–Sellers Energy Balance Climate Model with background information on the climate phenomena it models. We will also describe the range of equation parameters that reflect what can be found in the real world.

2.1 Climate Science Background

Energy balance, also described as the earth–atmosphere energy balance, is the balance between the incoming and outgoing energy from the Sun and the Earth, respectively [8]. From the energy emitted to the Earth from the Sun in the form of radiation, some is reflected by the clouds, but most is absorbed by the atmosphere and Earth's surface. The Earth also emits energy back to space in the form of infrared radiation, and it is the balance between this outgoing radiation and incoming solar energy that keeps Earth's surface at a habitable temperature and climate for life.

A key factor determining how much solar energy the Earth actually absorbs is albedo. Albedo (Figure 1) is the amount of sunlight reflected by surfaces, measured as a proportion between 0 and 1 [11]. A value of 0 means all incoming solar energy is absorbed, while a value of 1 means all of it is reflected. Light surfaces reflect a large amount of solar energy, which is high albedo, and darker surfaces absorb most solar energy, converting it into heat, which is low albedo. Albedo is therefore a critical determinant of how much solar energy the Earth actually absorbs. A small change in the distribution of light and dark surfaces across the planet can significantly shift the global energy balance. For example, fresh snow and ice have albedo values near 0.9, while open ocean has values as low as 0.06, meaning the presence vs absence of ice has an important effect on how much solar energy reaches the climate system.

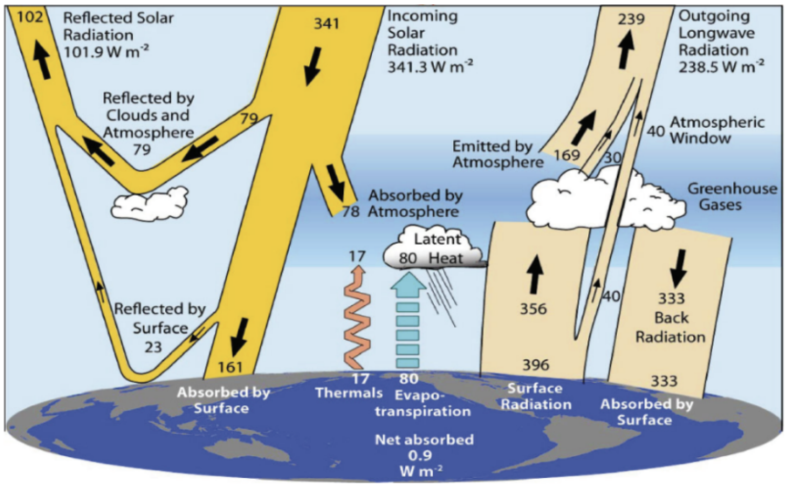

Figure 1. Albedo effect [6]

This sensitivity of albedo to surface conditions is what gives rise to one of the most significant feedback mechanisms in climate science. When this balance is disturbed, by a change in solar forcing or greenhouse gas concentrations, the ice-albedo feedback (Figure 2) can amplify the initial disruption. If the planet warms, ocean ice melts, exposing more ocean surface, which absorbs more heat and warms the planet further. On the other hand, if the planet cools, more ice can form which reflects more sunlight and cools the planet further. These self-reinforcing feedback loops are called the ice-albedo feedback [17].

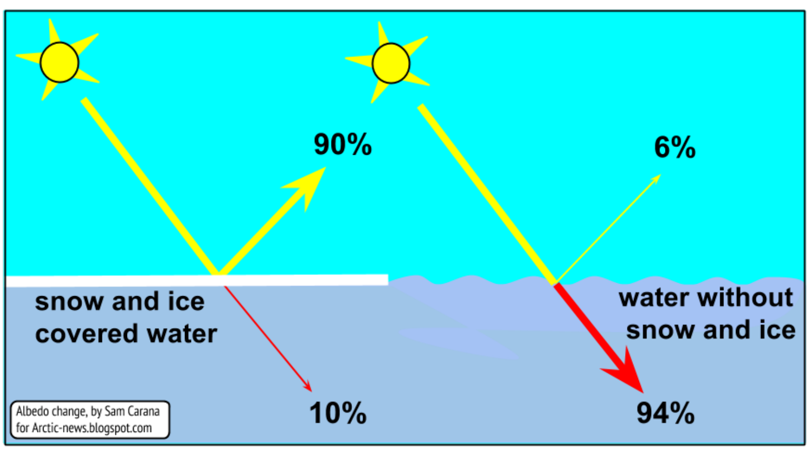

Figure 2. Ice-albedo feedback [1]

Because of these feedback loops, the climate can settle into two plausible stable states: (1) a warm state where ice is confined to the poles, albedo is low, and the planet absorbs a lot of solar energy; (2) a cold "Snowball Earth" state where ice covers most of the surface, albedo is high, and little solar energy is absorbed. The climate tipping point is the point of no return; if the Earth warms or cools to a certain extent, the system transitions between normal and these two states irreversibly [15]. Even if the original forcing is reversed, the system will not return without a much larger push in the opposite direction. Currently, greenhouse gas emissions make warming the most pressing threat.

To mathematically represent the climate and the associated ice-albedo effect, we will utilize a low-dimensional model, a simplified mathematical representation that uses a small number of variables [5]. Just physical intuition cannot tell us where the tipping points are or how stable each equilibrium is; for that, we need tools from dynamical systems. While these low-dimensional models are less accurate than full climate simulations, they isolate the key nonlinearity clearly, allowing formal analysis of the conditions under which tipping occurs.

2.2 Energy Balance Climate Model (EBCM)

The foundation of the Energy Balance Climate Model comes from the First Law of Thermodynamics: the conservation of energy [16]. Earth's climate system is driven by a continuous exchange of energy between the planet and space. To understand how the global average temperature is changing over time, we need to account for the net difference of the energy entering the system and the energy leaving it.

Energy enters the Earth system primarily as shortwave solar radiation from the Sun, but some is immediately reflected through albedo. The solar energy that is not reflected is absorbed by the Earth's surface and atmosphere, acting as the system's sole heating mechanism.

Conversely, the Earth loses energy by emitting longwave infrared radiation back into the vacuum of space. The amount of energy radiated away is highly dependent on the planet's surface temperature; as the Earth warms, it emits more energy. A stable climate equilibrium occurs when the absorbed incoming solar energy perfectly matches the outgoing infrared energy.

When these two energy flows are not perfectly equal, the Earth's temperature changes. The rate at which the temperature responds to this energy imbalance is governed by the system's heat capacity. Because the oceans and atmosphere can store large amounts of thermal energy, the climate does not change instantaneously but instead, the temperature gradually rises or falls until a new equilibrium is reached. This physical relationship allows us to define the central differential equation: the rate of change of temperature over time, scaled by the system's heat capacity, which is equal to the absorbed solar energy minus the outgoing radiation. This is the foundation of the Budyko-Sellers Energy Balance Climate Model, which represents the Earth's climate as a single global mean temperature T, making the essential dynamics of energy balance, ice-albedo feedback, and tipping behavior mathematically accessible.

3 Extension of Mathematical Concepts

3.1 Saddle-Node Bifurcation

In class, we analyzed autonomous first-order ordinary differential equations of the form

$$\frac{dx}{dt} = f(x),$$

where we focused on performing qualitative analysis. In particular, we solved for any existing equilibria, which are defined as solutions to

$$f(x) = 0,$$

with stability determined by the sign of $f(x)$ in neighboring regions.

To extend this concept, we introduce a parameter into the differential equation, leading to equations of the form

$$\frac{dx}{dt} = f(x; r),$$

where $r \in \mathbb{R}$ represents a system parameter. As a result, the number and stability of equilibria may change as r varies. This phenomenon is called a bifurcation.

A saddle-node bifurcation occurs when one stable equilibrium and one unstable equilibrium collide and annihilate each other as the parameter crosses a critical value. This extends the course concept of equilibria by allowing the existence of equilibria themselves to depend on a parameter.

Consider the example

$$\frac{dx}{dt} = r - x^2.$$

- If $r > 0$, there are two equilibria: $x = \pm\sqrt{r}$.
- If $r = 0$, there is one equilibrium: $x = 0$.
- If $r < 0$, there are no real equilibria.

Geometrically, a saddle-node bifurcation occurs when the graph of $f(x; r)$ becomes tangent to the horizontal axis. When the parameter value exceeds the critical threshold, the graph intersects the axis at two different points. When the parameter is at the critical value, the graph touches the axis. Lastly, when the value is below the threshold, there is no intersection with the axis.

$$f(x^*; r_c) = 0$$

and

$$\frac{\partial f}{\partial x}(x^*; r_c) = 0,$$

indicating a degenerate equilibrium where linear stability analysis is inconclusive.

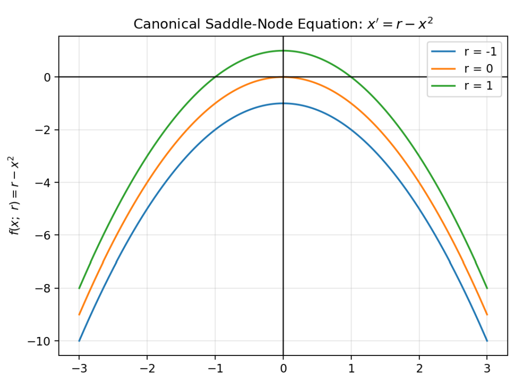

Figure 3. Example Saddle-Node Equation Diagram

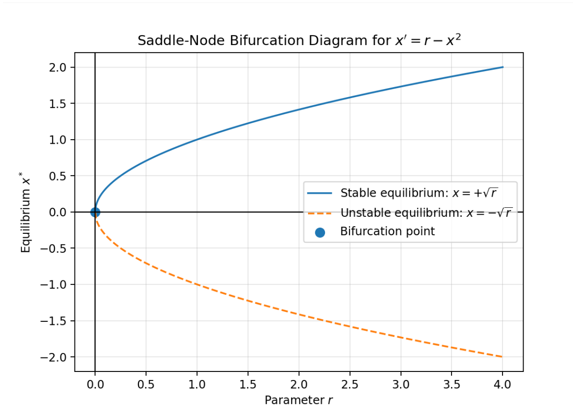

Figure 4. Example Bifurcation Diagram

An important consequence of saddle-node bifurcations is the emergence of threshold behavior. When the parameter reaches the critical value, the stable equilibrium becomes weaker, as indicated by

$$f'(x^*) \to 0.$$

At the bifurcation point, the equilibrium disappears completely. This forces solutions to transition to qualitatively different behavior, and this change reveals several underlying real-world phenomena in which gradual changes in external conditions lead to sudden shifts in system dynamics.

3.2 Hysteresis

When multiple saddle-node bifurcations are present, the system may exhibit hysteresis. Hysteresis is a phenomenon in which the long-term behavior of a system depends not only on the current parameter value but also on the path by which it has evolved.

In contrast to the phase line analysis developed in this course, which considers systems at fixed parameter values, hysteresis arises in parameter-dependent systems of the form

$$\frac{dx}{dt} = f(x; r),$$

where multiple stable equilibria may coexist over a range of parameter values.

As r varies, stable equilibria may disappear through saddle-node bifurcations. If r is increased, the system remains near a given stable equilibrium until that equilibrium disappears, at which point the solution suddenly shifts to another stable state. However, when r is decreased, the system does not immediately return to its original state. Instead, the reverse transition occurs at a different parameter value.

This asymmetry creates a hysteresis loop in the bifurcation diagram, where the forward and backward paths differ. Mathematically, this shows the existence of multiple attractors and the fact that transitions occur only when an equilibrium is destroyed, not when another equilibrium becomes favorable.

In applications such as Energy Balance Climate Models, hysteresis helps model abrupt and potentially irreversible transitions between climate states and offers a framework for understanding tipping points in complex systems.

4 Derivation of the Energy Balance Climate Model

4.1 Solar Radiation

The Earth's climate system's energy is sourced almost entirely from the Sun in the form of electromagnetic radiation. A commonly used approximation of the solar energy spectrum that reaches the Earth's atmosphere is a black body (a phenomenon where the radiation emitted by all materials are dependent on the object's temperature) at a temperature of $5780K$ [4].

The solar constant is the total solar irradiance, which equals the solar energy flux through a unit area of an imaginary sphere at the mean distance from Earth to the Sun, and is approximately 1368 W/m^2.

Insolation is the energy per unit area that reaches a specific location (usually on Earth's surface). At the top of Earth's atmosphere, the insolation is equal to the solar constant when Earth is at a mean distance from the Sun.

4.2 Energy Balance Models

Consider all components of the climate system that can exchange heat with outer space: all oceans, the entire atmosphere, and the soil of all land masses to a depth of several meters. As shown in Figure 1, different surfaces have varying albedo values.

At time t, the average temperature throughout this entire system is $T(t)$. We assume the heat capacity of this system is C, an average constant value. Let A be the surface area of the planet.

The amount of energy needed to reach a temperature is

$$AC\Delta T = T(t + \Delta t) = T(t) + \Delta T = A(E_{\text{in}} - E_{\text{out}})\Delta t,$$

where E_{in} is the amount of solar energy coming in, and E_{out} is the average amount of energy emitted by one square meter of Earth's surface per unit time.

By taking the limit as $\Delta t \to 0$:

$$C\frac{dT}{dt} = E_{\text{in}} - E_{\text{out}}.$$

Once the system reaches an equilibrium state, the Energy Balance Climate Model (EBCM) reduces to $E_{\text{in}} = E_{\text{out}}$.

4.3 Basic Model

When viewed from the Sun, the Earth is a flat disc with radius R (radius of the Earth). So, the amount of Solar Energy intercepted by Earth per unit time is $\pi R^2 S_o$, where S_o is the solar constant. As discussed in prior sections, the fraction of this energy reflected back into space is the albedo, denoted by α. Therefore, the amount of Solar Energy reaching Earth's surface per unit time is $(1-\alpha)\pi R^2$.

By this assumption, the solar energy is uniformly over the Earth's surface area, so when divided by $4\pi R^2$, the solar energy is $\frac{1}{4}(1-\alpha)S_o$. Let $Q = \frac{1}{4}S_o$, and thus $E_{in} = (1-\alpha)Q$.

The Earth emits electromagnetic radiation like the Sun, but its energy spectrum is almost entirely infrared. The amount of energy thus depends on temperature.

Assume Earth radiates as a black body with effective surface temperature T. Then average energy radiated out per unit area per unit time can be found using the Stefan-Boltzmann Law:

$$E_{out} : T \to E_{out}(T) = \sigma T^4.$$

By Stefan's Constant, σ is $5.67 * 10^{-8}$ W$m^{-2}K^{-4}$. The EBCM then reduces to $C\frac{dT}{dt} = (1-\alpha)Q - \sigma T^4$. At equilibrium, this becomes the energy balance equation:

$$(1-\alpha)Q = \sigma T^4$$

By solving for T, we find that $T^* is \left(\frac{(1-\alpha)Q}{\sigma}\right)^{1/4}$. Plugging in $\alpha = 0.30$ and $S_o = 1368$ W/m^2, Q is 342 W/m^2, and the basic model gives $T^* = 254.8K$.

However, the actual surface temperature is $287.7K$. This difference is largely due to the greenhouse effect of Earth's atmosphere, specifically the influence of gases like CO_2, water vapor, methane, and aerosols.

4.4 Greenhouse Effect

Greenhouse gases (CO_2, water vapor, methane) absorb infrared radiation, which reduces E_{out} but not E_{in}. Consequently, this raises Earth's equilibrium temperature.

To account for the greenhouse effect, we introduce an emissivity parameter ε where $0 < \varepsilon < 1$, which represents the fraction of infrared radiation that escapes to space after absorption by greenhouse gases. A value of $\varepsilon < 1$ reduces the outgoing radiation relative to a perfect blackbody, raising the equilibrium temperature. The outgoing radiation becomes $E_{\text{out}} = \varepsilon\sigma T^4$, where σ is the Stefan–Boltzmann constant. Setting $E_{\text{in}} = E_{\text{out}}$ with $E_{\text{in}} = (1-\alpha)Q$ results in the equilibrium temperature:

$$T^* = \left(\frac{(1-\alpha)Q}{\varepsilon\sigma}\right)^{1/4}.$$

With $\varepsilon = 0.62$, $\alpha = 0.30$, and $Q = 342$ W/m^2, this yields $T^* = 287.7$ K, matching the observed global mean surface temperature. Note that ε is fitted empirically rather than derived from first principles. In our final model we bypass this parameterization entirely, as Budyko's linearized outgoing radiation $E_{\text{out}} = A + BT$ is fit directly to observational data and implicitly absorbs the greenhouse effect through its coefficients.

4.5 Multiple Equilibria

So far, we have assumed that albedo α is a fixed constant. This assumption ignores a critical physical process: as temperatures drop, water freezes into ice and snow, which have much higher albedo than open ocean or land. To capture this, we replace the constant α with a temperature-dependent function.

A standard smooth approximation used in the literature is:

$$\alpha(T) = 0.5 - 0.2\tanh\left(\frac{T - 265}{10}\right),$$

where T is in Kelvin. This function transitions smoothly from high albedo ($\alpha \approx 0.7$) at low temperatures, corresponding to an ice-covered Earth, to low albedo ($\alpha \approx 0.3$) at high temperatures, corresponding to an ice-free Earth, with the transition centered near $T = 265$K.

With this substitution, E_{in} becomes a function of temperature:

$$E_{\text{in}}(T) = (1 - \alpha(T))Q.$$

Incorporating this into the EBCM with the greenhouse parameterization from the previous section, the governing ODE becomes:

$$C\frac{dT}{dt} = (1 - \alpha(T))Q - \varepsilon\sigma T^4.$$

At equilibrium, this reduces to:

$$(1 - \alpha(T))Q = \varepsilon\sigma T^4,$$

which can no longer be solved analytically for T. Graphically, the left-hand side $(1-\alpha(T))Q$ follows an S-shaped curve in T, while the right-hand side $\varepsilon\sigma T^4$ is uniformly increasing. For $\varepsilon = 0.6$, these two curves intersect at three points:

$$T_1^* \approx 288 \text{ K}, \qquad T_2^* \approx 265 \text{ K}, \qquad T_3^* \approx 233 \text{ K}.$$

The existence of three equilibria is a direct consequence of the nonlinearity introduced by the ice-albedo feedback. To determine stability, we examine the sign of $\frac{dT}{dt}$ in the regions between equilibria. Between T_3^* and T_2^*, the outgoing radiation exceeds the absorbed solar radiation, so $\frac{dT}{dt} < 0$ and solutions decrease toward the cold equilibrium T_3^*. Between T_2^* and T_1^*, the absorbed solar radiation exceeds the outgoing radiation, so $\frac{dT}{dt} > 0$ and solutions increase toward the warm equilibrium T_1^*. For temperatures below T_3^*, solutions increase toward T_3^*, while for temperatures above T_1^*, solutions decrease toward T_1^*. Thus T_2^* is unstable, while T_1^* and T_3^* are both stable.

The stable equilibrium $T_1^* \approx 288$ K corresponds to the current climate, with ice confined to the poles. The stable equilibrium $T_3^* \approx 233$ K corresponds to a “Snowball Earth” state, in which ice covers most of the planet’s surface and the high albedo traps the climate in a self-reinforcing cold state. The unstable equilibrium T_2^* acts as a threshold, where initial conditions above it are attracted to the warm state.

4.6 Budyko’s Model

So far, we have assumed that Earth radiates like a perfect blackbody, following the Stefan–Boltzmann Law $E_{\text{out}} = \sigma T^4$. In 1969, Budyko proposed replacing this theoretical expression with one fit directly to observational satellite data, yielding a simpler and empirically grounded model for outgoing radiation:

$$E_{\text{out}}(T) = A + BT,$$

where T is measured in Kelvin [6]. Converting the standard Budyko–Sellers parameterization from Celsius to Kelvin units gives the best-fit values $A_e = -367.58$ W/m^2 and $B = 2.09$ W m^{-2} K^{-1} for the current Northern Hemisphere climate. Because these constants are derived from real atmospheric data, the greenhouse effect is implicitly absorbed into them, removing the need for the separate emissivity parameter ε introduced in the previous section.

Substituting this expression into the EBCM alongside the temperature-dependent albedo $\alpha(T) = 0.5 - 0.2\tanh\left(\frac{T-265}{10}\right)$, the governing ODE becomes:

$$C\frac{dT}{dt} = (1 - \alpha(T))Q - (A + BT).$$

At equilibrium, $\frac{dT}{dt} = 0$, so the energy balance equation reduces to:

$$(1 - \alpha(T))Q = A + BT.$$

Graphically, this replaces the nonlinear Stefan–Boltzmann curve with a straight line, but the qualitative structure of the solution is unchanged. The S-shaped curve $(1 - \alpha(T))Q$ intersects the line $A + BT$ at three points, yielding two stable equilibria at the extremes and one unstable equilibrium in the middle. This is the final form of the EBCM that we analyze throughout the remainder of the paper.

5 Dynamical Analysis of the Energy Balance Climate Model

5.1 Multiple Climate States from Energy Balance

To explore the different climate states modeled through the Budyko–Sellers Energy Balance Climate Model (EBCM), we begin with the governing ordinary differential equation:

$$C\frac{dT}{dt} = Q(1 - \alpha(T)) - (A + BT), \tag{1}$$

where

- T is the global temperature in Kelvin,
- $A + BT$ represents outgoing infrared radiation,
- Q is the solar forcing parameter,
- $(1 - \alpha(T))$ is the absorbed solar energy.

For the numerical simulations, we use

$$Q = \frac{Q_0}{4} \approx 340.25 \text{ W/m}^2,$$

where $Q_0 \approx 1361\ \mathrm{W/m^2}$ is the modern solar constant measured from satellite observations [7]. The factor of $1/4$ converts incoming solar radiation into a global spherical average.

For the outgoing longwave radiation term, we use the Budyko linear approximation

$$I(T) = \widetilde{A} + BT,$$

with

$$\widetilde{A} = -367.58\ \mathrm{W/m^2}, \qquad B = 2.09\ \mathrm{W/m^2/K}.$$

These values are obtained by converting the standard Budyko–Sellers parameterization from Celsius to Kelvin units while preserving the same physical radiation balance [9].

Recall that our paper uses a smooth albedo function:

$$\alpha(T) = 0.5 - 0.2 \tanh\left(\frac{T - 265}{10}\right)$$

Finally, we use $C = 10\ \mathrm{W\,yr\,m^{-2}\,K^{-1}}$, which is consistent with typical mixed-layer ocean heat-capacity scales used in low-order Energy Balance Climate Models [12].

5.1.1 Existence and Uniqueness

Since $\alpha(T)$ is infinitely differentiable, the right-hand side of the differential equation is continuous and locally Lipschitz in T. Therefore, by the Existence and Uniqueness Theorem, there exists a unique solution $T(t)$ for any initial condition (T_0, t_0).

5.1.2 Equilibria

Equilibria occur when the temperature stops changing, so we set

$$f(T; Q) = Q(1 - \alpha(T)) - (A + BT) = 0. \tag{2}$$

Due to the nonlinearity of $\alpha(T)$, this equation cannot be solved explicitly. Instead, we qualitatively analyze the equilibria by identifying the intersections of the curves

$$Q(1 - \alpha(T)) \quad \text{and} \quad A + BT.$$

Because $\alpha(T)$ is nonlinear, the function $Q(1 - \alpha(T))$ is S-shaped, while $A + BT$ is linear. Depending on the value of Q, these curves may intersect at one or three points. However, for simplicity, we first set Q as a constant

$$min_{\mathrm{T}} Q(T) < Q < max_{\mathrm{T}} Q(T),$$

where the system has three equilibria: a stable cold equilibrium, an unstable intermediate equilibrium, and a stable warm equilibrium, as shown in Figure 4.

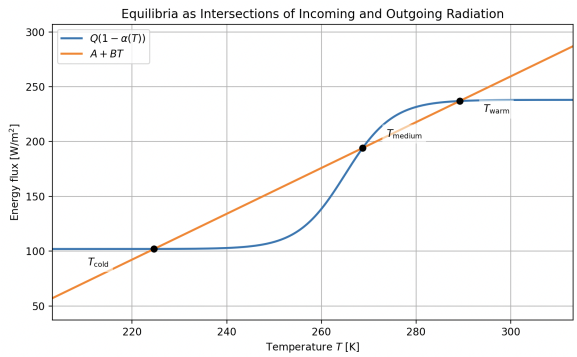

Figure 5. All Equilibrium States

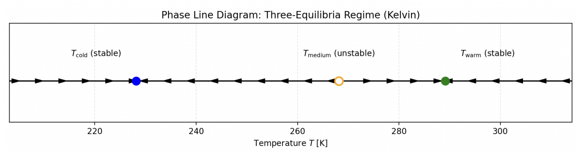

Figure 6. Phase Line Diagram: Three Equilibria Regime

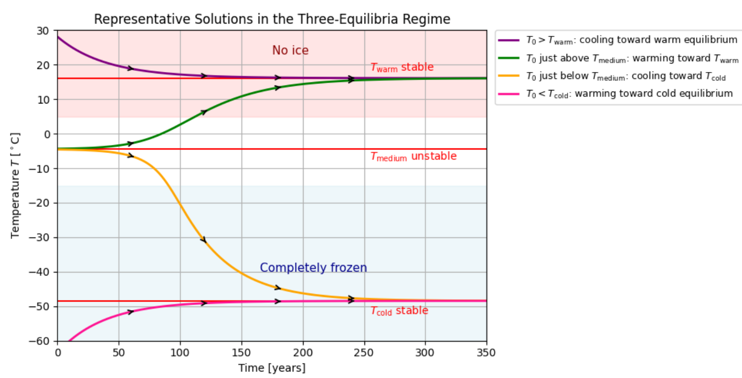

Figure 7. Solution Trajectories: Three Equilibria Regime

Thus, the system may have up to three equilibria:

$$T_{\text{cold}} < T_{\text{medium}} < T_{\text{warm}}.$$

The equilibrium T_{cold} represents a stable Snowball Earth state, where high albedo reflects most incoming radiation. Small perturbations are corrected, returning the system to this cold equilibrium.

The equilibrium T_{warm} represents a stable warm climate, where albedo is low and absorption is high. Again, perturbations are damped out.

Between these is the unstable equilibrium T_{medium}, where small perturbations are amplified. A slight increase in temperature reduces albedo and drives the system toward the warm state, while a slight decrease pushes the system toward the Snowball state. Therefore, T_{medium} acts as a tipping point.

5.1.3 The Potential Function

Beyond identifying equilibria graphically, we can visualize the dynamics of the EBCM by constructing a potential function $V(T)$ such that the system descends V over time. Following the standard construction of potential functions for one-dimensional autonomous systems [14], define

$$V(T) = -\int f(T)\,dT,$$

where $f(T) = Q(1 - \alpha(T)) - (A + BT)$.

Since the Budyko ODE is

$$C\frac{dT}{dt} = f(T),$$

we have

$$\frac{dT}{dt} = \frac{1}{C}f(T).$$

Therefore,

$$\frac{dV}{dt} = V'(T)\frac{dT}{dt} = -f(T)\cdot\frac{1}{C}f(T) = -\frac{1}{C}f(T)^2 \leq 0.$$

Thus, V is non-increasing along trajectories. Equilibria of the ODE correspond to critical points of V: stable equilibria are local minima, and unstable equilibria are local maxima.

Computing the integral explicitly using $\int \tanh(u)\,du = \ln\cosh(u)$, we obtain

$$V(T) = -(0.5Q - A)T - 2Q\ln\cosh\left(\frac{T - 265}{10}\right) + \frac{B}{2}T^2.$$

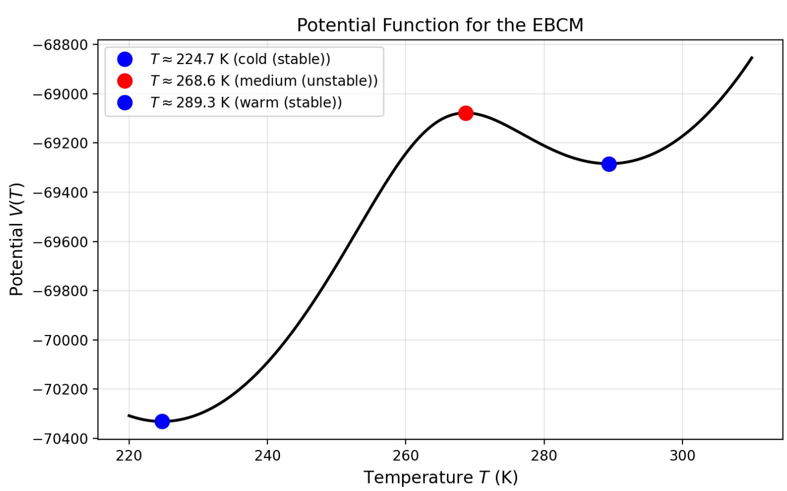

Figure 8. Potential Function $V(T)$ Showing Double-Well Structure

The figure shows a characteristic double-well structure. The two local minima of V correspond to the stable equilibria from above $T_{cold} \approx 225K$ and $T_{warm} \approx 289K$, and the local maximum in the middle corresponds to the unstable equilibrium $T_{medium} \approx 269K$. This picture provides a mechanical and visual representation of the climate system. Temperature acts like a rolling ball on the surface of $V(T)$. A small perturbation will keep the ball in its local valley, while a large perturbation will push the ball over the "hill" at T_{medium} into the opposite valley. This is the geometric origin of the climate tipping behavior, as the unstable equilibrium in the middle acts as a barrier between the two opposing climate regimes.

5.2 Regimes as Q Varies

We examine how the number of equilibria changes as the parameter Q varies. Since equilibria satisfy

$$Q(1-\alpha(T)) - (A+BT) = 0,$$

we can solve for Q as a function of T:

$$Q(T) = \frac{A+BT}{1-\alpha(T)}.$$

the function $Q(T)$ determines the number of equilibria through its extrema.

Thus, the regimes can be described as follows:

- If $Q < \min_T Q(T)$, then only the Snowball Earth equilibrium exists, as shown in Figure 8.

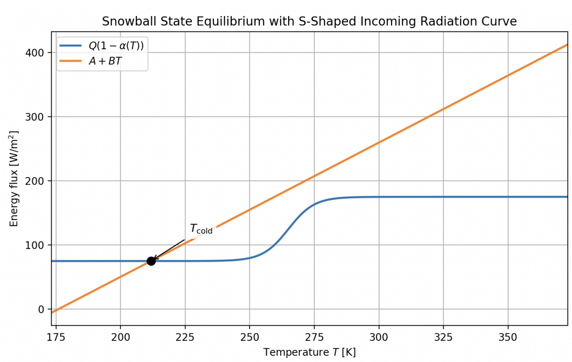

Figure 9. Snowball Earth State Equilibrium

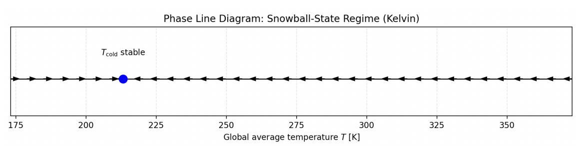

Figure 10. Phase Line Diagram: Snowball Earth State Regime

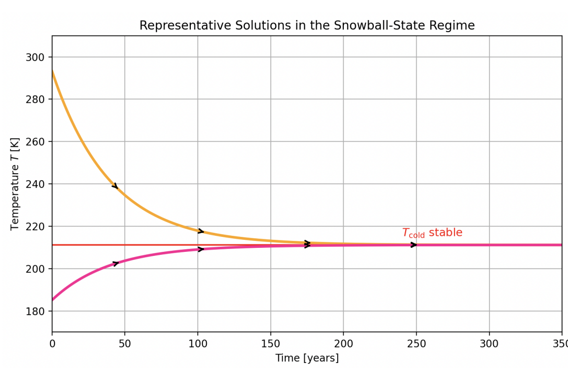

Figure 11. Solution Trajectories: Snowball Earth-State

In this case, the incoming and outgoing radiation curves intersect at only one point, corresponding to the stable cold equilibrium T_{cold}. The warm and intermediate equilibria disappear through saddle-node bifurcations, leaving the fully glaciated state as the only equilibrium configuration.

The phase line diagram shows that trajectories move toward T_{cold} from both sides, confirming that the equilibrium is stable. The representative solution trajectories display the same behavior dynamically: solutions beginning at relatively warm temperatures cool toward the Snowball Earth state, while colder initial conditions gradually warm upward toward the same equilibrium. Consequently, every trajectory converges to the cold stable state over time.

Physically, this regime relates to a planet with extensive ice coverage and high planetary albedo. Because a large fraction of incoming solar radiation is reflected back into space, the climate system cannot sustain a warm equilibrium. The positive ice–albedo feedback therefore locks the system into a globally frozen state.

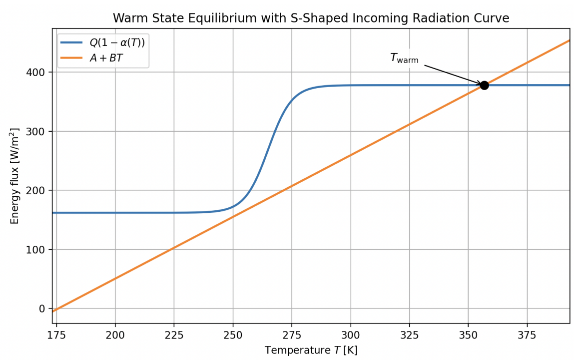

Figure 12. Warm State Equilibrium

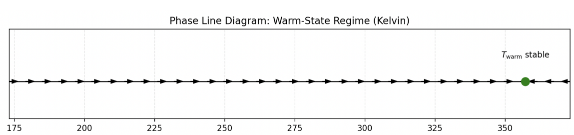

Figure 13. Phase Line Diagram: Warm State Regime

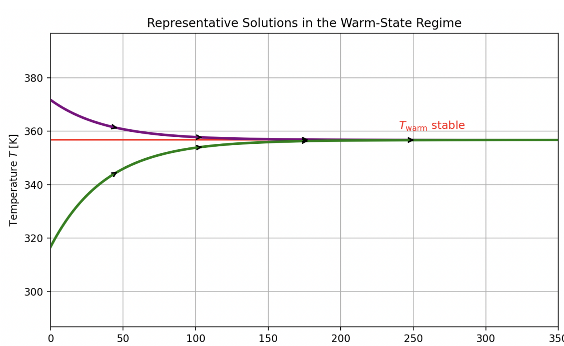

Figure 14. Solution Trajectories: Warm-State Regime

- If $Q > \max_T Q(T)$, then only the warm equilibrium exists, as shown in Figure 12.

In this parameter range, the incoming radiation curve intersects the outgoing radiation curve at only one point, corresponding to the stable warm equilibrium T_{warm}. Since no cold or intermediate equilibria exist, all solution trajectories evolve toward the same warm climate state regardless of initial temperature.

The phase line diagram confirms the stability of this equilibrium: arrows point toward T_{warm} from both directions, indicating that perturbations decay over time. The representative solution trajectories further demonstrate this behavior. Initial conditions above the equilibrium cool downward toward T_{warm}, while initial conditions below the equilibrium warm upward toward the same state. Thus, the warm equilibrium acts as a global attractor in this regime.

Physically, this corresponds to a climate with relatively low albedo and strong absorption of incoming solar radiation. Because ice coverage is limited, the ice–albedo feedback reinforces warming and prevents the existence of a competing Snowball Earth state.

Together, these figures demonstrate how varying the parameter Q changes the number and type of equilibria in the system, highlighting the role of nonlinear feedbacks in producing multiple climate states.

5.2.1 Bifurcation Structure

We now consider how equilibria evolve as Q varies. This behavior is captured by the bifurcation diagram, which plots equilibrium temperature T^* as a function of the parameter Q.

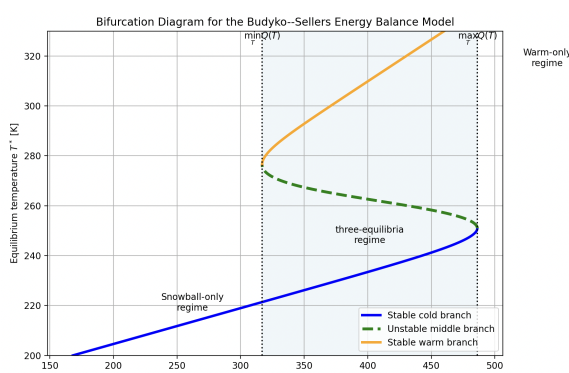

Figure 15. Bifurcation Diagram of EBCM

The resulting diagram has a characteristic S-shaped curve (Figure 15). For intermediate values of Q, three branches of equilibria are present: a lower branch corresponding to the stable cold (Snowball Earth) state, an upper branch corresponding to the stable warm climate, and a middle branch corresponding to an unstable equilibrium. The stable branches are typically drawn as solid curves, while the unstable branch is represented by a dashed curve.

As Q decreases, the upper stable branch and the unstable middle branch approach each other and eventually merge at a critical value of Q. Similarly, as Q increases, the lower stable branch and the unstable branch merge at another critical value. At these points, the curve folds over itself, creating two turning points in the diagram.

These turning points correspond to saddle-node bifurcations, where a stable and unstable equilibrium collide and disappear. Outside of these critical values, the diagram contains only a single equilibrium branch: the cold branch for sufficiently small Q, and the warm branch for sufficiently large Q.

This S-shaped structure reflects the nonlinear dependence of the albedo function on temperature and is responsible for the existence of multiple equilibria and tipping points in the system.

5.3 Calculating the Hysteresis Window

Recall that the equilibrium condition of the EBCM satisfies

$$Q(T) = \frac{A + BT}{1 - \alpha(T)}.$$

In this paper, recall that we use

$$\alpha(T) = 0.5 - 0.2\tanh(\frac{T - 265}{10})$$

as the smooth albedo function reliant on temperature and use the values $\widetilde{A} = -367.58$ W/m^2 and $B = 2.09$ W/m^2/K.

To find $\min_T Q(T)$ and $\max_T Q(T)$, which are the fold bifurcations of the S-curve, mathematically calculate where $\frac{dQ}{dT} = 0$.

Using the quotient rule,

$$\frac{dQ}{dT} = \frac{B(1 - \alpha(T)) - (\tilde{A} + BT)(-\alpha'(T))}{(1 - \alpha(T))^2} = 0.$$

$\frac{dQ}{dT}$ is equal to 0 when

$$B(1 - \alpha(T)) - (\tilde{A} + BT)(-\alpha'(T)) = 0,$$

where

$$\alpha'(T) = -0.2 \cdot \frac{1}{10}\text{sech}^2\left(\frac{T - 265}{10}\right) = -0.02\text{sech}^2\left(\frac{T - 265}{10}\right).$$

Plugging in the constants, we get

$$2.09(0.5 - 0.2\tanh(\frac{T - 265}{10})) - (-367.58 + 2.09T)(-0.02\text{sech}^2\left(\frac{T - 265}{10}\right)) = 0.$$

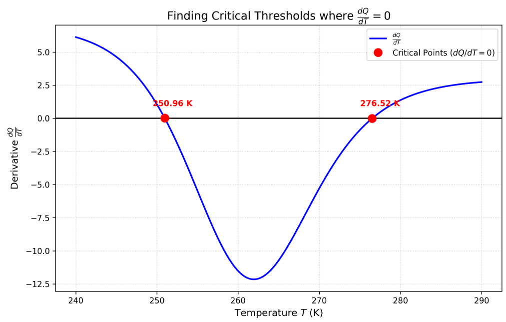

Figure 16. Finding Critical Thresholds where $\frac{dQ}{dT} = 0$

The local maximum of $Q(T)$ occurs near $T \approx 250.96\,\mathrm{K}$, while the local minimum occurs near $T \approx 276.52\,\mathrm{K}$.

Plugging our critical temperatures into $Q(T)$, we get

$$Q(250.96) = \frac{-367.58 + 2.09(250.96)}{1 - [0.5 - 0.2\tanh(\frac{250.96-265}{10})]} = \frac{156.93}{0.3228} = 486.15W/m^2$$

and

$$Q(276.52) = \frac{-367.58 + 2.09(276.52)}{1 - [0.5 - 0.2\tanh(\frac{276.52-265}{10})]} = \frac{210.3468}{0.663683} = 316.94W/m^2$$

(Note: W/m^2 is Watts per square meter)

The difference between these two values is:

$$\Delta Q = \max_T Q(T) - \min_T Q(T) \approx 169.21W/m^2,$$

which is the hysteresis window. This means that if solar forcing drops to $316.9W/m^2$ and triggers the Snowball Earth phase, it cannot be undone by simply restoring Q to $316.9W/m^2$. Instead, Q must be pushed to $486.2W/m^2$, which is $169.21W/m^2$ higher than where Snowball Earth is triggered. The system recalls which state it fell into and requires more energy to escape, representing hysteresis.

6 Interpretation

From the mathematical structure of the Energy Balance Climate Model, we see that the Earth's climate systems are governed by nonlinear feedback loops and thresholds. The concept of bifurcation is at the center of this model; when solar forcing (Q) and temperature (T) cross critical values, the current stable equilibrium can immediately lose stability and move to a drastically different state. This provides the physical basis for understanding climate anomaly hypotheses, such as Snowball Earth. In that scenario, the ice-albedo feedback spirals out of control: once ice covers enough of the surface, the albedo term in the ODE drops so significantly that the planet is mathematically locked in a state of total ice. Similarly, these models provide clarity for interpreting other potential states, like an unending warm state that many climate scientists are warning about. Although modern climate science relies on high-dimensional simulations, these simple ODE-based low-dimensional models are still essential to isolate the fundamental physics of climate tipping and provide cheaper estimations of hypothetical scenarios. They illustrate that climate change is not a reversible process; once a tipping point is reached, simply returning parameters to their original values may not be enough to recover the previous climate state, highlighting the risks of pushing the Earth system beyond its limits.

7 Discussion

7.1 Limitations

It is important to note that while the original Budyko–Sellers framework is celebrated for its inclusion of a latitude parameter, this paper focuses on the zero-dimensional (0D) global energy balance — meaning that we only have one parameter for global temperature, without any x and y values. By treating the Earth as having a uniform global mean temperature, we simplify many climate phenomena for an analysis relevant to our course level. This approach provides a clearer view of the fundamental bifurcation structure and the different hypotheses of Snowball Earth and warm state, even with the sacrifice of the regional detail that a full latitudinal study would provide. We also utilize a smoothed albedo function for a continuously differentiable right hand side (and also a consequence of our low-dimensional model). Due to stark differences between the boundaries of ice, ocean, and other surfaces, an albedo function that captures those details is a discontinuous function. We opt for a smoothed function so we can use the same tools we learned in class: qualitative analysis, apply saddle-node bifurcation theorem, and more. We made these decisions for simplicity and relevance to course level so we can still introduce these topics to our peers despite the sacrifices.

7.2 Future Direction

A potential future direction for our project group is to upgrade the model to a one-dimensional version. Instead of treating the Earth as a single point, we would incorporate the position so the temperature parameter would become $T(y, t)$. This would be the logical next step to increase the accuracy of the model by resolving the spatial heterogeneity in temperature and albedo across the planet from utilizing a single global mean. However, this will expand the complexity of the math by incorporating more partial differential equations.

8 Conclusion

In this report, we studied climate models based on the balance of the Earth–atmosphere system. We were able to derive the Budyko–Sellers Energy Balance Climate Model:

$$C\frac{dT}{dt} = (1 - \alpha(T))Q - (A + BT),$$

using the Law of Conservation of Energy. This ODE outlines the sensitivity of Earth's climate phenomena to changes in solar energy and ice cover and how feedback loops can cause the climate to shift between extremes. Then, we calculated accurate parameter values for the Energy Balance Climate Model and simulated calculations to better understand concepts such as hysteresis, ice-albedo feedback, Snowball Earth, and the Warm State equilibrium. Altogether, these results helped broaden our knowledge of climate dynamics and the mathematics driving investigations of climate change.

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9 Appendix

9.1 Statement on AI

Our group utilized AI for preliminary research and to generate Python code for the graphs and diagrams based off the equations we sourced from research.

Preliminary research was used as a starting point to find initial sources. Although we found the Budyko-Sellers Energy Balance Climate Model through manual research, we used AI as an assistive tool to develop a higher-level contextual understanding of the model. From then on, we conducted rigorous research and mathematical procedures on our own. Furthermore, AI was not used to write our project. All content was originally written by our group members, with the exception of receiving some grammatical help from AI.

AI was also used to generate Python code. We used Matplotlib to create graphs critical to our qualitative analysis (equilibria graphs, phase diagrams, solution trajectories, a potential function graph, and a bifurcation diagram) in Section 5, as well as two graphs (an equilibria graph and a bifurcation diagram) for a canonical example in Section 3. The code generated for these diagrams are included below.

Minor uses of AI include questions regarding LaTeX formatting, such as how to represent mathematical notation. For the paper's overall outline, however, we followed a pre-existing template that was not AI-generated. We did not use AI to translate plain text into LaTeX.

9.2 Code

Figure 3

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-3, 3, 500)
r_values = [-1, 0, 1]

plt.figure(figsize=(7, 5))

for r in r_values:
    f = r - x**2
    plt.plot(x, f, label=f"r = {r}")
```

```
plt.axhline(0, color="black", linewidth=1)
plt.axvline(0, color="black", linewidth=1)
plt.title("Canonical Saddle-Node Equation: $x' = r - x^2$")
plt.xlabel("x")
plt.ylabel("$f(x;\\ r) = r - x^2$")
plt.legend()
plt.grid(True, alpha=0.3)
plt.show()
```

Figure 4

```
import numpy as np
import matplotlib.pyplot as plt

r = np.linspace(0, 4, 500)
stable_branch = np.sqrt(r)
unstable_branch = -np.sqrt(r)

plt.figure(figsize=(7, 5))

plt.plot(r, stable_branch, label="Stable equilibrium: $x = +\\sqrt{r}$")
plt.plot(r, unstable_branch, linestyle="--", label="Unstable equilibrium: $x = -\\sqrt{r}$"
plt.scatter([0], [0], s=60, label="Bifurcation point")

plt.axhline(0, color="black", linewidth=1)
plt.axvline(0, color="black", linewidth=1)
plt.title("Saddle-Node Bifurcation Diagram for $x' = r - x^2$")
plt.xlabel("Parameter $r$")
plt.ylabel("Equilibrium $x^*$")
plt.legend()
plt.grid(True, alpha=0.3)
plt.show()
```

Figure 5

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
Q = 340
A = -367.5   # Kelvin-adjusted intercept: 203.3 - 2.09*273.15
B = 2.09

# Temperature range in Kelvin
T = np.linspace(203.15, 313.15, 2000)  # equivalent to -70°C to 40°C
```

```python
# Smooth albedo function in Kelvin
def alpha(T):
    return 0.5 - 0.2 * np.tanh((T - 265) / 10)

# Curves
incoming = Q * (1 - alpha(T))
outgoing = A + B * T
f = incoming - outgoing

# Find equilibria
equilibria = []

for i in range(len(T) - 1):
    if f[i] * f[i + 1] < 0:
        T_eq = (T[i] + T[i + 1]) / 2
        equilibria.append(T_eq)

equilibria = sorted(equilibria)

# Assign labels
labels = []

if len(equilibria) == 3:
    labels = [
        r"$T_{\mathrm{cold}}$",
        r"$T_{\mathrm{medium}}$",
        r"$T_{\mathrm{warm}}$"
    ]
elif len(equilibria) == 1:
    if equilibria[0] > 273.15:
        labels = [r"$T_{\mathrm{warm}}$"]
    else:
        labels = [r"$T_{\mathrm{cold}}$"]

# Plot
fig, ax = plt.subplots(figsize=(8, 5))

ax.plot(T, incoming, label=r"$Q(1-\alpha(T))$", linewidth=2)
ax.plot(T, outgoing, label=r"$A+BT$", linewidth=2)

# Label equilibria
for i, T_eq in enumerate(equilibria):
    y_eq = A + B * T_eq
    ax.plot(T_eq, y_eq, "ko", markersize=6)
```

```
    if i == 0:
        dx, dy = -14, -15
    elif i == 1 and len(equilibria) == 3:
        dx, dy = 5, 10
    else:
        dx, dy = 5, -10

    ax.text(
        T_eq + dx,
        y_eq + dy,
        labels[i],
        fontsize=10,
        bbox=dict(facecolor="white", alpha=0.6, edgecolor="none")
    )

# Formatting
ax.set_xlim(203.15, 313.15)
ax.set_ylim(
    min(np.min(incoming), np.min(outgoing)) - 20,
    max(np.max(incoming), np.max(outgoing)) + 20
)

ax.set_xlabel(r"Temperature $T$ [K]")
ax.set_ylabel(r"Energy flux [W/m$^2$]")
ax.set_title("Equilibria as Intersections of Incoming and Outgoing Radiation")

ax.grid(True)
ax.legend()

plt.tight_layout()
plt.show()
```

Figure 6

```
import numpy as np
import matplotlib.pyplot as plt

# Convert to Kelvin
T_cold = -45 + 273.15
T_medium = -5 + 273.15
T_warm = 16 + 273.15

# Plot setup
fig, ax = plt.subplots(figsize=(10, 2.5))
ax.axhline(0, color='black')
```

```python
xmin = T_cold - 25
xmax = T_warm + 25

# Uniform arrow spacing in Kelvin
arrow_positions = np.linspace(xmin, xmax, 25)

# Direction field (phase line)
for xi in arrow_positions:
    if xi < T_cold:
        direction = 1
    elif T_cold < xi < T_medium:
        direction = -1
    elif T_medium < xi < T_warm:
        direction = 1
    else:
        direction = -1

    dx = 2 * direction

    ax.arrow(
        xi, 0, dx, 0,
        head_width=0.05,
        head_length=1.5,
        fc='black',
        ec='black',
        length_includes_head=True
    )

# Equilibria points
ax.plot(T_cold, 0, 'o', color='blue', markersize=10)
ax.plot(T_medium, 0, 'o', markerfacecolor='white',
        markeredgecolor='orange', markersize=10, markeredgewidth=2)
ax.plot(T_warm, 0, 'o', color='green', markersize=10)

# Labels
ax.text(T_cold - 3, 0.3, r"$T_{\mathrm{cold}}$ (stable)", ha='right')
ax.text(T_medium, 0.3, r"$T_{\mathrm{medium}}$ (unstable)", ha='center')
ax.text(T_warm + 3, 0.3, r"$T_{\mathrm{warm}}$ (stable)", ha='left')

# Formatting
ax.set_xlim(xmin, xmax)
ax.set_ylim(-0.5, 0.7)
ax.set_yticks([])

ax.set_xlabel(r"Temperature $T$ [K]")
ax.set_title("Phase Line Diagram: Three-Equilibria Regime (Kelvin)")
```

```
ax.grid(True, linestyle="--", alpha=0.3)

plt.tight_layout()
plt.show()
```

Figure 7

```
import numpy as np
import matplotlib.pyplot as plt

# Model parameters
A = 203.3
B = 2.09
Q = 340
C = 80

# Smooth albedo function
def alpha(T_celsius):
    T_kelvin = T_celsius + 273.15
    return 0.5 - 0.2 * np.tanh((T_kelvin - 265) / 10)

def net_energy(T):
    return Q * (1 - alpha(T)) - (A + B * T)

def dTdt(T):
    return net_energy(T) / C

# Find equilibria numerically
def find_equilibria(Tmin=-70, Tmax=40, N=5000):
    Ts = np.linspace(Tmin, Tmax, N)
    Fs = np.array([net_energy(T) for T in Ts])

    roots = []

    for i in range(len(Ts) - 1):
        if Fs[i] * Fs[i + 1] < 0:
            a, b = Ts[i], Ts[i + 1]

            for _ in range(60):
                m = (a + b) / 2
                if net_energy(a) * net_energy(m) <= 0:
                    b = m
                else:
                    a = m
```

```python
            roots.append((a + b) / 2)

    return roots

equilibria = find_equilibria()

if len(equilibria) != 3:
    raise ValueError(f"Expected 3 equilibria, but found {len(equilibria)}: {equilibria}")

T_cold, T_medium, T_warm = equilibria

print("Computed equilibria:")
print(f"T_cold   = {T_cold:.3f} °C")
print(f"T_medium = {T_medium:.3f} °C")
print(f"T_warm   = {T_warm:.3f} °C")

# RK4 solver
def solve_ode(T0, t):
    T = np.zeros_like(t)
    T[0] = T0
    dt = t[1] - t[0]

    for i in range(len(t) - 1):
        k1 = dTdt(T[i])
        k2 = dTdt(T[i] + 0.5 * dt * k1)
        k3 = dTdt(T[i] + 0.5 * dt * k2)
        k4 = dTdt(T[i] + dt * k3)

        T[i + 1] = T[i] + (dt / 6) * (k1 + 2*k2 + 2*k3 + k4)

    return T

# Representative solutions
t = np.linspace(0, 350, 1200)

T_above_warm = solve_ode(T_warm + 12, t)
T_between_medium_warm = solve_ode(T_medium + 0.1, t)
T_between_cold_medium = solve_ode(T_medium - 0.1, t)
T_below_cold = solve_ode(T_cold - 15, t)

# Add arrows
def add_arrows(ax, t, T, spacing=60):
    arrow_times = np.arange(spacing, t[-1] - spacing, spacing)

    for at in arrow_times:
        i = np.argmin(np.abs(t - at))
```

```python
        j = min(i + 8, len(t) - 1)

        ax.annotate(
            "",
            xy=(t[j], T[j]),
            xytext=(t[i], T[i]),
            arrowprops=dict(arrowstyle="->", linewidth=1.3, color="black"),
        )

# Plot
fig, ax = plt.subplots(figsize=(8, 5))

# Shaded regions
ax.fill_between([0, 350], y1=5, y2=30, color="red", alpha=0.10)
ax.fill_between([0, 350], y1=-60, y2=-15, color="lightblue", alpha=0.20)

ax.text(175, 23, "No ice", color="darkred", fontsize=11)
ax.text(165, -40, "Completely frozen", color="darkblue", fontsize=11)

# Equilibrium lines
ax.axhline(T_warm, color="red", linewidth=1.4)
ax.axhline(T_medium, color="red", linewidth=1.4)
ax.axhline(T_cold, color="red", linewidth=1.4)

# Representative curves
ax.plot(t, T_above_warm, color="purple", linewidth=2)
ax.plot(t, T_between_medium_warm, color="green", linewidth=2)
ax.plot(t, T_between_cold_medium, color="orange", linewidth=2)
ax.plot(t, T_below_cold, color="deeppink", linewidth=2)

# Arrows
add_arrows(ax, t, T_above_warm)
add_arrows(ax, t, T_between_medium_warm)
add_arrows(ax, t, T_between_cold_medium)
add_arrows(ax, t, T_below_cold)

# Labels
ax.text(255, T_warm + 2.0, r"$T_{\mathrm{warm}}$ stable", color="red", fontsize=10)
ax.text(255, T_medium - 3.2, r"$T_{\mathrm{medium}}$ unstable", color="red", fontsize=10)
ax.text(255, T_cold - 4.0, r"$T_{\mathrm{cold}}$ stable", color="red", fontsize=10)

# Formatting
ax.set_xlim(0, 350)
ax.set_ylim(-60, 30)
ax.set_xlabel("Time [years]")
ax.set_ylabel(r"Temperature $T$ [$^\circ$C]")
```

```python
ax.set_title("Representative Solutions in the Three-Equilibria Regime")
ax.grid(True)

plt.show()
```

Figure 8

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import brentq

# parameters
Q = 340.25        # solar forcing (W/m^2)
A = -367.58       # outgoing radiation intercept (Kelvin convention)
B = 2.09          # outgoing radiation slope (W/m^2/K)

# albedo function
def alpha(T):
    return 0.5 - 0.2 * np.tanh((T - 265) / 10)

# right-hand side of the ODE
def f(T):
    return Q * (1 - alpha(T)) - (A + B * T)

# potential function
def V(T):
    return -(0.5 * Q - A) * T - 2 * Q * np.log(np.cosh((T - 265) / 10)) + (B / 2) * T**2

# temperature range
T_vals = np.linspace(220, 310, 1000)
V_vals = V(T_vals)
f_vals = f(T_vals)

# find equilibria
equilibria = []
for i in range(len(T_vals) - 1):
    if f_vals[i] * f_vals[i+1] < 0:  # sign change
        root = brentq(f, T_vals[i], T_vals[i+1])
        equilibria.append(root)

print("Equilibria found at:", equilibria)

# plot
fig, ax = plt.subplots(figsize=(8, 5))
ax.plot(T_vals, V_vals, color='black', linewidth=2)
```

```python
# mark equilibria
labels = ['cold (stable)', 'medium (unstable)', 'warm (stable)']
colors = ['blue', 'red', 'blue']
for i, T_eq in enumerate(equilibria):
    label = labels[i] if i < len(labels) else f'eq {i+1}'
    color = colors[i] if i < len(colors) else 'green'
    ax.plot(T_eq, V(T_eq), 'o', color=color, markersize=10,
            label=f'$T \\approx {T_eq:.1f}$ K ({label})')

ax.set_xlabel('Temperature $T$ (K)', fontsize=12)
ax.set_ylabel('Potential $V(T)$', fontsize=12)
ax.set_title('Potential Function for the EBCM', fontsize=13)
ax.legend(loc='best', fontsize=10)
ax.grid(True, alpha=0.3)

plt.tight_layout()
plt.savefig('potential_function.png', dpi=200)
plt.show()
```

Figure 9

```python
import numpy as np
import matplotlib.pyplot as plt

A = -367.5   # Kelvin-adjusted: 203.3 - 2.09*273.15
B = 2.09
Q = 250      # low Q chosen to show snowball-only regime

# Temperature range in Kelvin
T = np.linspace(173.15, 373.15, 20000)  # equivalent to -100°C to 100°C

def alpha(T):
    return 0.5 - 0.2 * np.tanh((T - 265) / 10)

incoming = Q * (1 - alpha(T))
outgoing = A + B * T
f = incoming - outgoing

# Find intersections
equilibria = []

for i in range(len(T) - 1):
    if f[i] * f[i + 1] < 0:
        T_eq = (T[i] + T[i + 1]) / 2
        equilibria.append(T_eq)
```

```python
if len(equilibria) == 0:
    raise ValueError("No equilibrium found. Increase T range or adjust Q.")

T_eq = equilibria[0]
y_eq = A + B * T_eq

fig, ax = plt.subplots(figsize=(8, 5))

ax.plot(T, incoming, label=r"$Q(1-\alpha(T))$", linewidth=2)
ax.plot(T, outgoing, label=r"$A+BT$", linewidth=2)

# Mark intersection
ax.plot(T_eq, y_eq, "ko", markersize=8)

ax.annotate(
    r"$T_{\mathrm{cold}}$",
    xy=(T_eq, y_eq),
    xytext=(T_eq + 15, y_eq + 45),
    arrowprops=dict(arrowstyle="->"),
    fontsize=11,
    bbox=dict(facecolor="white", alpha=0.85, edgecolor="none")
)

# Formatting: full graph shows
ax.set_xlim(173.15, 373.15)
ax.set_ylim(
    min(np.min(incoming), np.min(outgoing)) - 20,
    max(np.max(incoming), np.max(outgoing)) + 20
)

ax.set_xlabel(r"Temperature $T$ [K]")
ax.set_ylabel(r"Energy flux [W/m$^2$]")
ax.set_title("Snowball State Equilibrium with S-Shaped Incoming Radiation Curve")

ax.grid(True)
ax.legend()

plt.tight_layout()
plt.show()

print("Equilibrium (K):", T_eq)
print("Equilibrium (°C):", T_eq - 273.15)
```

Figure 10

```python
import numpy as np
```

```python
import matplotlib.pyplot as plt

T_cold = -60 + 273.15  # 213.15 K

# Plot setup
fig, ax = plt.subplots(figsize=(10, 2.5))
ax.axhline(0, color="black", linewidth=1.2)

x_min = 173.15
x_max = 373.15

arrow_positions = np.linspace(x_min, x_max, 35)

# Direction field
for xi in arrow_positions:
    if xi < T_cold:
        dx = 3      # move right toward equilibrium
    else:
        dx = -3     # move left toward equilibrium

    ax.arrow(
        xi, 0, dx, 0,
        head_width=0.05,
        head_length=2,
        fc="black",
        ec="black",
        length_includes_head=True
    )

# Equilibrium point
ax.plot(T_cold, 0, "o", color="blue", markersize=10)

# Label
ax.text(T_cold, 0.32, r"$T_{\mathrm{cold}}$ stable",
        ha="center", fontsize=10)

# Formatting
ax.set_xlim(x_min, x_max)
ax.set_ylim(-0.5, 0.7)
ax.set_yticks([])

ax.set_xlabel(r"Global average temperature $T$ [K]")
ax.set_title("Phase Line Diagram: Snowball-State Regime (Kelvin)")

ax.grid(True, linestyle="--", alpha=0.3)
```

```
plt.tight_layout()
plt.show()
```

Figure 11

```
import numpy as np
import matplotlib.pyplot as plt

# Time domain
t = np.linspace(0, 350, 1200)

# Convert equilibrium to Kelvin
T_cold = 211.15    # = -62 °C + 273.15

# Representative solutions
def approach_equilibrium(t, T0, Teq, rate=0.025):
    return Teq + (T0 - Teq) * np.exp(-rate * t)

# Initial conditions in Kelvin
T_above = approach_equilibrium(t, 293.15, T_cold, rate=0.025)  # 20 °C
T_below = approach_equilibrium(t, 185.15, T_cold, rate=0.025)  # -88 °C

# Plot
fig, ax = plt.subplots(figsize=(8, 5))

# Equilibrium line
ax.axhline(T_cold, color="red", linewidth=1.5)

# Solution trajectories
ax.plot(t, T_above, color="orange", linewidth=2.5)
ax.plot(t, T_below, color="deeppink", linewidth=2.5)

# Direction arrows
def add_arrows(T):
    for i in [150, 350, 600, 850]:
        ax.annotate(
            "",
            xy=(t[i+8], T[i+8]),
            xytext=(t[i], T[i]),
            arrowprops=dict(
                arrowstyle="->",
                color="black",
                lw=1.5
            )
        )
```

```
add_arrows(T_above)
add_arrows(T_below)

# Label equilibrium
ax.text(
    240,
    T_cold + 4,
    r"$T_{\mathrm{cold}}$ stable",
    color="red",
    fontsize=11
)

# Formatting
# Wider y-range so the full curves are visible
ax.set_xlim(0, 350)
ax.set_ylim(170, 310)

ax.set_xlabel("Time [years]")
ax.set_ylabel(r"Temperature $T$ [K]")

ax.set_title(
    "Representative Solutions in the Snowball-State Regime"
)

ax.grid(True)

plt.tight_layout()
plt.show()
```

Figure 12

```
import numpy as np
import matplotlib.pyplot as plt

A = -367.5    # Kelvin-adjusted: 203.3 - 2.09*273.15
B = 2.09
Q = 540       # warm-only regime

# Kelvin range, equivalent to -100°C to 120°C
T = np.linspace(173.15, 393.15, 20000)

def alpha(T):
    return 0.5 - 0.2 * np.tanh((T - 265) / 10)

incoming = Q * (1 - alpha(T))
outgoing = A + B * T
```

```python
f = incoming - outgoing

# Find intersection
equilibria = []

for i in range(len(T) - 1):
    if f[i] * f[i + 1] < 0:
        T_eq = (T[i] + T[i + 1]) / 2
        equilibria.append(T_eq)

if len(equilibria) == 0:
    raise ValueError("No equilibrium found. Increase T range or adjust Q.")

T_eq = equilibria[-1]
y_eq = A + B * T_eq

# Plot
fig, ax = plt.subplots(figsize=(8, 5))

ax.plot(T, incoming, label=r"$Q(1-\alpha(T))$", linewidth=2)
ax.plot(T, outgoing, label=r"$A+BT$", linewidth=2)

# Mark intersection
ax.plot(T_eq, y_eq, "ko", markersize=8)

ax.annotate(
    r"$T_{\mathrm{warm}}$",
    xy=(T_eq, y_eq),
    xytext=(T_eq - 40, y_eq + 40),
    arrowprops=dict(arrowstyle="->"),
    fontsize=11,
    bbox=dict(facecolor="white", alpha=0.85, edgecolor="none")
)

# Formatting: full graph visible
ax.set_xlim(173.15, 393.15)
ax.set_ylim(
    min(np.min(incoming), np.min(outgoing)) - 20,
    max(np.max(incoming), np.max(outgoing)) + 20
)

ax.set_xlabel(r"Temperature $T$ [K]")
ax.set_ylabel(r"Energy flux [W/m$^2$]")
ax.set_title("Warm State Equilibrium with S-Shaped Incoming Radiation Curve")

ax.grid(True)
```

```
ax.legend()

plt.tight_layout()
plt.show()

print("Equilibrium (K):", T_eq)
print("Equilibrium (°C):", T_eq - 273.15)
```

Figure 13

```
import numpy as np
import matplotlib.pyplot as plt

T_warm = 84.0 + 273.15   # 357.15 K

# Plot setup
fig, ax = plt.subplots(figsize=(10, 2.5))
ax.axhline(0, color="black", linewidth=1.2)

# Kelvin domain (equivalent to -100°C to 100°C)
x_min = 173.15
x_max = 373.15

arrow_positions = np.linspace(x_min, x_max, 35)

# Direction field
for xi in arrow_positions:
    if xi < T_warm:
        dx = 3      # move right toward equilibrium
    else:
        dx = -3     # move left toward equilibrium

    ax.arrow(
        xi, 0, dx, 0,
        head_width=0.05,
        head_length=2,
        fc="black",
        ec="black",
        length_includes_head=True
    )

# Equilibrium point
ax.plot(T_warm, 0, "o", color="green", markersize=10)

# Label
ax.text(T_warm, 0.32, r"$T_{\mathrm{warm}}$ stable",
```

```
        ha="center", fontsize=10)

# Formatting
ax.set_xlim(x_min, x_max)
ax.set_ylim(-0.5, 0.7)
ax.set_yticks([])

ax.set_xlabel(r"Temperature $T$ [K]")
ax.set_title("Phase Line Diagram: Warm-State Regime (Kelvin)")

ax.grid(True, linestyle="--", alpha=0.3)

plt.tight_layout()
plt.show()
```

Figure 14

```
import numpy as np
import matplotlib.pyplot as plt

# Model parameters
A = 203.3
B = 2.09
Q = 540
C = 80

# Smooth albedo
def alpha(T):
    T_K = T + 273.15
    return 0.5 - 0.2 * np.tanh((T_K - 265) / 10)

def dTdt(T):
    return (Q * (1 - alpha(T)) - (A + B*T)) / C

# Find warm equilibrium
def find_equilibrium():
    T_vals = np.linspace(-50, 150, 5000)
    f_vals = dTdt(T_vals)

    for i in range(len(T_vals)-1):
        if f_vals[i] * f_vals[i+1] < 0:
            a, b = T_vals[i], T_vals[i+1]
            for _ in range(50):
                m = (a + b) / 2
                if dTdt(a) * dTdt(m) <= 0:
                    b = m
```

```
                else:
                    a = m
            return (a + b)/2

T_warm = find_equilibrium()
print("T_warm =", T_warm)

# Solve ODE
def solve(T0, t):
    T = np.zeros_like(t)
    T[0] = T0
    dt = t[1] - t[0]

    for i in range(len(t)-1):
        T[i+1] = T[i] + dt * dTdt(T[i])

    return T

# Time grid
t = np.linspace(0, 350, 1000)

# Start NEAR equilibrium (fixes weird S-shape)
T_above = solve(T_warm + 15, t)
T_below = solve(T_warm - 40, t)

# Plot
fig, ax = plt.subplots(figsize=(8,5))

# Equilibrium line
ax.axhline(T_warm, color="red", linewidth=1.5)

# Trajectories
ax.plot(t, T_above, color="purple", linewidth=2)
ax.plot(t, T_below, color="green", linewidth=2)

# Arrows
def add_arrows(T):
    for i in range(150, 800, 200):
        ax.annotate(
            "",
            xy=(t[i+5], T[i+5]),
            xytext=(t[i], T[i]),
            arrowprops=dict(arrowstyle="->", color="black", lw=1.2)
        )

add_arrows(T_above)
```

```
add_arrows(T_below)

# Labels
ax.text(240, T_warm + 3, r"$T_{\mathrm{warm}}$ stable", color="red", fontsize=11)

# Formatting
ax.set_xlim(0, 350)
ax.set_ylim(0, 200)   # ← your requested interval

ax.set_xlabel("Time [years]")
ax.set_ylabel("Temperature T [°C]")
ax.set_title("Representative Solutions in the Warm-State Regime")

ax.grid(True)

plt.show()
```

Figure 15

```
import numpy as np
import matplotlib.pyplot as plt

# Model parameters
A = 203.3
B = 2.09

# Temperature range
T = np.linspace(-100, 100, 10000)

# Smooth albedo function
def alpha(T_celsius):
    T_kelvin = T_celsius + 273.15
    return 0.5 - 0.2 * np.tanh((T_kelvin - 265) / 10)

# Equilibrium condition:
# Q(1 - alpha(T)) = A + BT
# Solve for Q as a function of T

Q_of_T = (A + B * T) / (1 - alpha(T))

# Find turning points
dQdT = np.gradient(Q_of_T, T)

turning_points = []

for i in range(len(T) - 1):
```

```python
    if dQdT[i] * dQdT[i + 1] < 0:
        T_turn = (T[i] + T[i + 1]) / 2
        Q_turn = (A + B * T_turn) / (1 - alpha(T_turn))
        turning_points.append((T_turn, Q_turn))

# Sort by Q value
turning_points = sorted(turning_points, key=lambda x: x[1])

(T_low, Q_low), (T_high, Q_high) = turning_points

print(f"Lower critical point: Q = {Q_low:.2f}, T = {T_low:.2f} °C")
print(f"Upper critical point: Q = {Q_high:.2f}, T = {T_high:.2f} °C")

# Stability classification
# Stable branches: lower cold branch and upper warm branch
# Unstable branch: middle branch between turning points

stable_cold = T < T_low
unstable_middle = (T >= T_low) & (T <= T_high)
stable_warm = T > T_high

# Plot bifurcation diagram
fig, ax = plt.subplots(figsize=(8, 5))

# Plot Q on x-axis and equilibrium temperature on y-axis
ax.plot(Q_of_T[stable_cold], T[stable_cold],
        color="blue", linewidth=2, label="Stable cold branch")

ax.plot(Q_of_T[unstable_middle], T[unstable_middle],
        color="green", linestyle="--", linewidth=2, label="Unstable middle branch")

ax.plot(Q_of_T[stable_warm], T[stable_warm],
        color="orange", linewidth=2, label="Stable warm branch")

# Critical Q values
ax.axvline(Q_low, color="black", linestyle=":", linewidth=1.5)
ax.axvline(Q_high, color="black", linestyle=":", linewidth=1.5)

ax.text(Q_low, 90, r"$\min_T Q(T)$", ha="center", fontsize=10)
ax.text(Q_high, 90, r"$\max_T Q(T)$", ha="center", fontsize=10)

# Shaded three-equilibria region
ax.fill_betweenx(
    [-100, 100],
    Q_low,
    Q_high,
```

```
    color="lightblue",
    alpha=0.20
)

ax.text(
    (Q_low + Q_high) / 2,
    -30,
    "three-equilibria\nregime",
    ha="center",
    fontsize=10
)

# Labels for one-equilibrium regimes
ax.text(Q_low - 65, -85, "Snowball-only\nregime", ha="center", fontsize=10)
ax.text(Q_high + 65, 70, "Warm-only\nregime", ha="center", fontsize=10)

# Formatting
ax.set_xlabel(r"Solar forcing parameter $Q$")
ax.set_ylabel(r"Equilibrium temperature $T^*$ [$^\circ$C]")
ax.set_title("Bifurcation Diagram for the Budyko--Sellers Energy Balance Model")
ax.set_ylim(-100, 100)
ax.grid(True)
ax.legend(loc="lower right")

plt.show()
```